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if x is less than $7^\circ 15'$, then to five decimals, $\log_{10} x = \log_{10} \sin x + \frac{1}{3} \log_{10} \sec x = \log_{10} \tan x - \frac{2}{3} \log_{10} \sec x$.

SOLUTION BY THE PROPOSER.

$$y = \log \frac{x}{\sin x}, \text{ or } e^y = \frac{x}{\sin x}; \quad \frac{dy}{dx} = \frac{1}{x} - \cot x,$$

$$z = \log \sec x, \text{ or } e^z = \sec x; \quad \frac{dz}{dx} = \tan x.$$

Hence,

$$\frac{dy}{dx} = \tan x \frac{dy}{dz}.$$

Also

$$z - y = \log \frac{\tan x}{x}, \text{ or } e^{z-y} = \frac{\tan x}{x}.$$

$$\tan x \frac{dy}{dz} = \frac{1}{x} - \cot x; \quad \tan^2 x \frac{dy}{dz} = \frac{\tan x}{x} - 1, \quad \frac{dy}{dz} = \frac{e^{z-y} - 1}{e^z - 1} = \frac{e^{-y} - e^{-z}}{e^z - e^{-z}}.$$

$$z = 0 = y \quad \text{when} \quad x = 0.$$

$$\left. \frac{dy}{dz} \right|_{z=0} = 0 = \frac{-\frac{dy}{dz} \big|_{z=0} e^{-y} + e^{-z}}{e^z + e^{-z}} \bigg|_{z=0}; \quad \left. \frac{dy}{dz} \right|_{z=0} = \frac{-\frac{dy}{dz} \big|_{z=0} + 1}{2}; \quad \left. \frac{dy}{dz} \right|_{z=0} = \frac{1}{3}.$$

$$\left. \frac{d^2y}{dz^2} \right|_{z=0} = \frac{1}{(e^z + e^{-z})^2} \left[2 - e^{-y} \left([e^z - e^{-z}] \frac{dy}{dz} + [e^z + e^{-z}] \right) \right]; \quad \left. \frac{d^2y}{dz^2} \right|_{z=0} = 0.$$

$$\begin{aligned} \left. \frac{d^2y}{dz^2} \right|_{z=0} &= \frac{1}{2(e^{2z} - e^{-2z})} \left[-e^{-y} \left[(e^z - e^{-z}) \frac{d^2y}{dz^2} - (e^z - e^{-z}) \left(\frac{dy}{dz} \right)^2 + (e^z - e^{-z}) \right] \right]_{z=0} \\ &= \frac{-e^{-y}}{2(e^z + e^{-z})} \left[\frac{d^2y}{dz^2} - \left(\frac{dy}{dz} \right)^2 + 1 \right]_{z=0}; \quad \left. \frac{d^2y}{dz^2} \right|_{z=0} = -\frac{8}{45}. \end{aligned}$$

$$y = \frac{z}{3} - \frac{8}{45} z^2 \cdots, \quad z - y = \frac{2}{3} z + \frac{8}{45} z^2 \cdots,$$

or

$$\log x - \log \sin x = \frac{1}{3} \log \sec x - \frac{8}{45} (\log \sec x)^2,$$

$$\log_{10} x - \log_{10} \sin x = \frac{1}{3} \log \sec x - \frac{8}{45M} (\log \sec x)^2.$$

Similarly,

$$\log_{10} \tan x - \log_{10} x = \frac{2}{3} \log \sec x + \frac{8}{45M} (\log \sec x)^2,$$

$$\frac{8}{45M} (\log_{10} \sec x)^2 = .000005 \text{ if } \log_{10} \sec x = .0034948, x = 7^\circ 15',$$

$$\frac{8}{45M} = .4 \text{ nearly, } = (.40926), \quad = \frac{70}{171} \text{ very closely.}$$

This theorem enables one to dispense without inconvenience with the usual table for small angles. It is well to take $\log_{10} \text{ c.m. } 1' = 6.46372\frac{2}{3} - 10$ in order to account for odd thirds. The theorem gives correct results in most instances for values of θ up to about 10° . For $\theta = 10^\circ$ there is an error of one unit in the fifth place of decimals.

369. Proposed by I. A. BARNETT, Chicago, Ill.

Compute the definite integral $\int_a^b \log x dx$ by direct summation.

I. SOLUTION BY A. M. HARDING, University of Arkansas.

Let

$$S_n = dx \cdot \log a + dx \cdot \log(a + dx) + dx \cdot \log(a + 2dx) + \cdots + dx \cdot \log[a + (n-1)dx].$$

Then

$$I = \int_a^b \log x dx = \lim_{n \rightarrow \infty} S_n, \text{ where } ndx = b - a.$$

Now

$$\log(a + dx) = \log a + \frac{1}{a} dx - \frac{1}{2a^2} dx^2 + \frac{1}{3a^3} dx^3 - \frac{1}{4a^4} dx^4 + \cdots$$

$$\log(a + 2dx) = \log a + \frac{1}{a} 2dx - \frac{1}{2a^2} 2^2 dx^2 + \frac{1}{3a^3} 2^3 dx^3 - \frac{1}{4a^4} 2^4 dx^4 + \cdots$$

$$\log[a + (n-1)dx] = \log a + \frac{1}{a} (n-1)dx - \frac{1}{2a^2} (n-1)^2 dx^2 + \frac{1}{3a^3} (n-1)^3 dx^3 - \frac{1}{4a^4} (n-1)^4 dx^4 + \cdots$$

Hence,

$$\begin{aligned} S_n &= \log a \cdot ndx + \frac{1}{a} [1 + 2 + 3 + 4 + \cdots + (n-1)] dx^2 - \frac{1}{2a^2} [1^2 + 2^2 + 3^2 + \cdots + (n-1)^2] dx^3 \\ &\quad + \frac{1}{3a^3} [1^3 + 2^3 + 3^3 + \cdots + (n-1)^3] dx^4 - \frac{1}{4a^4} [1^4 + 2^4 + 3^4 + \cdots + (n-1)^4] dx^5 + \cdots \\ &= (b-a) \log a + \frac{1}{a} (n^2/2 - n/2) dx^2 - \frac{1}{2a^2} (n^3/3 - n^2/2 + n/6) dx^3 \\ &\quad + \frac{1}{3a^3} (n^4/4 - n^3/2 + n^2/4) dx^4 - \frac{1}{4a^4} (n^5/5 - n^4/2 + n^3/3 - n/30) dx^5 \\ &\quad + \frac{1}{5a^5} (n^6/6 - n^5/2 + 5n^4/12 - n^2/12) dx^6 - \cdots; \end{aligned}$$

$$\begin{aligned} \therefore I &= \lim S_n = (b-a) \log a + \frac{(b-a)^2}{2a} - \frac{(b-a)^3}{2 \cdot 3a^2} + \frac{(b-a)^4}{3 \cdot 4a^3} - \frac{(b-a)^5}{4 \cdot 5a^4} + \frac{(b-a)^6}{5 \cdot 6a^5} - \cdots \\ &= (b-a) \log a + \left(1 - \frac{1}{2}\right) \frac{(b-a)^2}{a} - \left(\frac{1}{2} - \frac{1}{3}\right) \frac{(b-a)^3}{a^2} + \left(\frac{1}{3} - \frac{1}{4}\right) \frac{(b-a)^4}{a^3} \\ &\quad + \left(\frac{1}{4} - \frac{1}{5}\right) \frac{(b-a)^5}{a^4} - \cdots \\ &= (b-a) \log a + (b-a) \left[- (1-b/a) - 1/2(1-b/a)^2 - 1/3(1-b/a)^3 - 1/4(1-b/a)^4 - \cdots \right] \\ &\quad + a \left[-\frac{1}{2} \left(\frac{b-a}{a}\right)^2 + \frac{1}{3} \left(\frac{b-a}{a}\right)^3 - \frac{1}{4} \left(\frac{b-a}{a}\right)^4 + \frac{1}{5} \left(\frac{b-a}{a}\right)^5 - \cdots \right] \\ &= (b-a) \log a + (b-a) \log \left[1 - \left(1 - \frac{b}{a}\right) \right] + a \left[\log \left\{ 1 - \left(1 - \frac{b}{a}\right) \right\} + \frac{b-a}{a} \right] \\ &= (b-a) \log a + (b-a) \log \frac{b}{a} + a \log \frac{b}{a} + b - a \\ &= (b-a) \log a + b \log \frac{b}{a} + b - a \\ &= b \log b - a \log a + b - a. \end{aligned}$$

II. SOLUTION BY THE PROPOSER.



Using the method of division indicated in the diagram, we have,

$$q^n = b \quad \text{and} \quad n = \log \frac{b}{a} - \log q.$$

Hence,

$$\begin{aligned}
 \int_a^b \log x dx &= \lim_{q \rightarrow 1} [a(q-1) \log a + aq(q-1) \log aq + \cdots + aq^{n-1}(q-1) \log aq^{n-1}], \\
 &= \lim_{q \rightarrow 1} a(q-1)[(1+q+\cdots+q^{n-1}) \log a + (q+2q^2+\cdots+(n-1)q^{n-1}) \log q], \\
 &= \lim_{q \rightarrow 1} a(q-1) \left[\left(\frac{q^n-1}{q-1} \right) \log a + \left\{ \frac{q-nq^n+(n-1)q^{n+1}}{(1-q)^2} \right\} \log q \right]^*, \\
 &= \lim_{q \rightarrow 1} a \left[(q^n-1) \log a + \frac{q(1-q^n)}{q-1} \log q + nq^n \log q \right], \\
 &= \lim_{q \rightarrow 1} a \left[\left(\frac{b-a}{a} \right) \log a + \frac{q}{q-1} \left(\frac{a-b}{a} \right) \log q + \frac{b}{a} \log \frac{b}{a} \right], \\
 &= b \log b - a \log a + b - a.
 \end{aligned}$$

Also solved similarly by P. PENALVER.

MECHANICS.

292. Proposed by C. N. SCHMALL, New York City.

In a bombardment, a battleship directs its fire at a fort standing on a hill whose height is a feet above sea level. The angle of elevation of the fort is found to be ϕ . If the initial velocity of the projectile is v , show that the fort will *not* be struck if $v < \sqrt{ag(1 + \csc \phi)}$.

SOLUTION BY PAUL CAPRON, Annapolis, Maryland.

If the projectile is fired at an elevation θ , its trajectory may be represented by $x = vt \cos \theta$, $y = vt \sin \theta - \frac{1}{2}gt^2$, or by

$$y = x \tan \theta - \frac{gx^2}{2v^2} \sec^2 \theta.$$

To find the envelope of all such trajectories, we have

$$0 = x \sec^2 \theta - \frac{gx^2}{v^2} \sec^2 \theta \tan \theta, \quad \text{or} \quad \tan \theta = \frac{v^2}{gx};$$

whence

$$y = \frac{v^2}{2g} - \frac{gx^2}{2v^2}$$

is the envelope.

* To sum

$$\begin{aligned}
 q + 2q^2 + 3q^3 + \cdots + (n-1)q^{n-1} &= q(1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2}) = q\Sigma \\
 \Sigma &= 1 + 2q + 3q^2 + \cdots + (n-1)q^{n-2} \\
 - q\Sigma &= -q - 2q^2 - \cdots - (n-2)q^{n-2} - (n-1)q^{n-1} \\
 \hline
 \therefore (1-q)\Sigma &= 1 + q + q^2 + \cdots + q^{n-2} - (n-1)q^{n-1} \\
 &= \frac{q^{n-1}-1}{q-1} - (n-1)q^{n-1} \\
 \therefore q\Sigma &= \frac{q-nq^n+(n-1)q^{n+1}}{(1-q)^2}.
 \end{aligned}$$